Advance
KENTUCKY

AP Calculus Prep Session Handout

Derivatives and Graphs of Functions

Derivatives and Graphs of Functions

The relationship between the graph of a function and its first and second derivatives appears often on the AP exams. It will appear on the free response section, often with the graph of f' given. It has appeared often in combination with the Fundamental Theorem of Calculus by defining a function as a definite integral and providing information, usually a graph, of the integrand. This handout avoids questions where the Fundamental Theorem of Calculus is needed.

Most of these questions would require you to justify your answers. A justification should appeal to a known calculus test or theorem. You have to show/state that the hypotheses are true and then draw the correct conclusion. Possible justifications include:

- If $f' > 0$ on an interval, then f is increasing on that interval.
- If $f' < 0$ on an interval, then f is decreasing on that interval.
- If f' is increasing on an interval, then the graph of f is concave up on that interval.
- If f' is decreasing on an interval, then the graph of f is concave down on that interval.
- The First Derivative Test
  - If the sign of f' changes from positive to negative at $x = c$, then f has a relative maximum at $x = c$.
  - If the sign of f' changes from negative to positive at $x = c$, then f has a relative minimum at $x = c$.
- The Second Derivative Test
  - If $f'(c) = 0$ and $f"(c) > 0$, then $f(c)$ is a relative minimum.
  - If $f'(c) = 0$ and $f"(c) < 0$, then $f(c)$ is a relative maximum.

Included here are several recent questions since this topic has shown up very often. It is rare that this question is asked where the students are expected to compute derivatives; often the question is graphing calculator neutral.

What Students Should Be Able to Do:

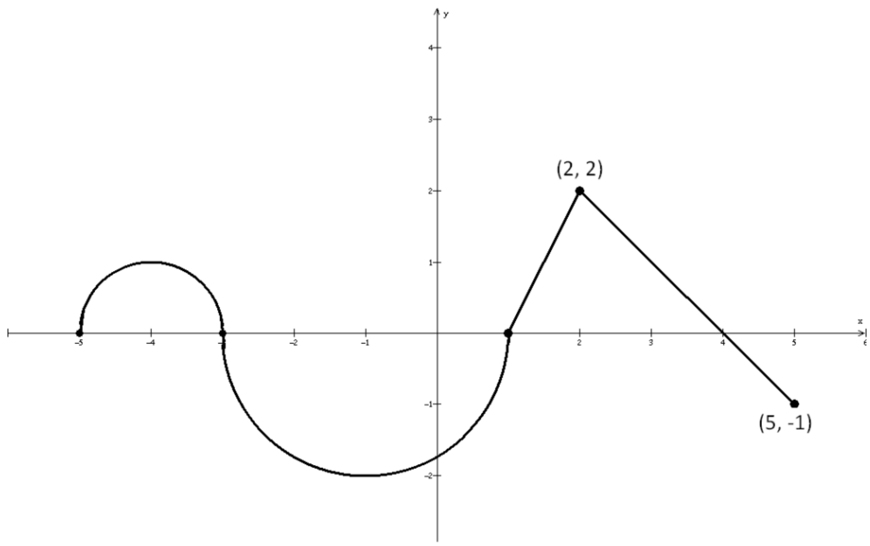
You need to be able to

- Determine whether a function is increasing/decreasing given its derivative.
- Determine whether the concavity of a function's graph given its derivative.
- Locate a function's relative and absolute extrema given its derivative.
- Locate a function's point(s) of inflection given its derivative.
- Reason from a graph without finding an explicit closed function that represents the graph.
- Write justifications and explanations.

Free Response Questions

2007 AB 4 Form B

No calculator.



Graph of f'

Let f be a function defined on the closed interval $-5 \le x \le 5$ with $f(1) = 3$. The graph of f', the derivative of f, consists of two semicircles and two line segments, as shown above.

(a) For $-5 < x < 5$, find all values of x at which f has a relative maximum. Justify your answer.

(b) For $-5 < x < 5$, find all values of x at which the graph of f has a point of inflection. Justify your answer.

(c) Find all intervals on which the graph of f is concave up and also has a positive slope. Explain your reasoning.

(d) Find the absolute minimum value of $f(x)$ over the closed interval $-5 \le x \le 5$. Explain your reasoning.

2001 AB 4 and BC 4

No calculator.

Let *h* be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of *h* is given by $h'(x) = \frac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of *x* for which the graph of *h* has a horizontal tangent, and determine whether *h* has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of *h* concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of *h* at *x* = 4.

(d) Does the line tangent to the graph of *h* at *x* = 4 lie above or below the graph of *h* for $x > 4$? Why?

2001 AB 5

No calculator.

A cubic polynomial function *f* is defined by

$$f(x) = 4x^3 + ax^2 + bx + k$$

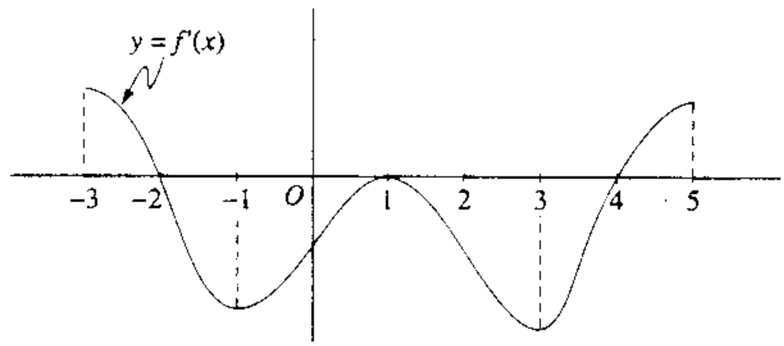
where *a*, *b*, and *k* are constants. The function *f* has a local minimum at $x = -1$, and the graph of *f* has a point of inflection at $x = -2$.

(a) Find the values of *a* and *b*.

(b) If $\int_0^1 f(x)\,dx = 32$, what is the value of *k*?

1996 AB 1

Graphing calculator allowed.



Note: This is the graph of f', not the graph of *f*.

The figure above shows the graph of f', the derivative of a function *f*. The domain of *f* is the set of all real numbers *x* such that $-3 < x < 5$.

(a) For what values of *x* does *f* have a relative maximum? Why?

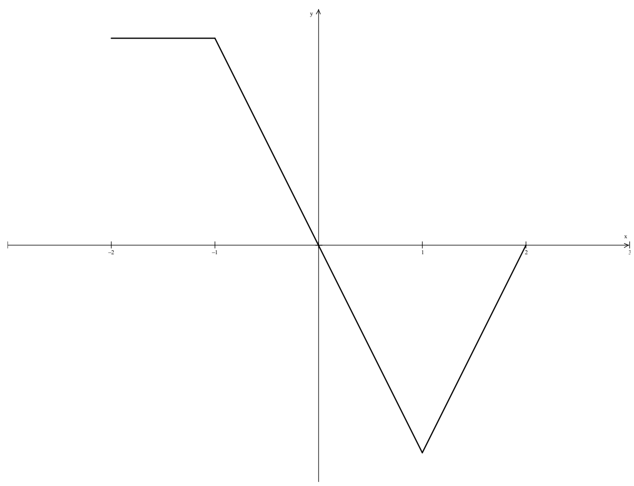
(b) For what values of *x* does *f* have a relative minimum? Why?

(c) On what intervals is the graph of *f* concave upward? Use f' to justify your answer.

(d) Suppose that $f(1) = 0$. Draw a sketch that shows the general shape of the graph of the function *f* on the open interval $0 < x < 2$.

Multiple Choice Questions

Part A. No calculator.



Graph of f'

1. The graph of f', the derivative of the function f, is shown above. Which of the following statements is true about f?
   (A) f is decreasing for $-1 \le x \le 1$.
   (B) f is increasing for $-2 \le x \le 0$.
   (C) f is increasing for $1 \le x \le 2$.
   (D) f has a local minimum at $x = 0$.
   (E) f is not differentiable at $x = -1$ and $x = 1$.

2. Let f be the function with derivative given by $f'(x) = x^2 - \frac{2}{x}$. On which of the following intervals is f decreasing?
   (A) $(-\infty, -1]$ only
   (B) $(-\infty, 0)$
   (C) $[-1, 0)$ only
   (D) $(0, \sqrt[3]{2}]$
   (E) $[\sqrt[3]{2}, \infty)$

3. Let f be the function given by $f(x) = 2xe^x$. The graph of f is concave down when
   (A) $x < -2$ (B) $x > -2$ (C) $x < -1$ (D) $x > -1$ (E) $x < 0$

x	-4	-3	-2	-1	0	1	2	3	4
$g'(x)$	2	3	0	-3	-2	-1	0	3	2

4. The derivative g' of a function g is continuous and has exactly two zeros. Selected values of g' are given in the table above. If the domain of g is the set of all real numbers, then g is decreasing on which of the following intervals?

   (A) $-2 \leq x \leq 2$ only

   (B) $-1 \leq x \leq 1$ only

   (C) $x \geq -2$

   (D) $x \geq 2$ only

   (E) $x \leq -2$ or $x \geq 2$

5. The function f is given by $f(x) = x^4 + x^2 - 2$. On which of the following intervals is f increasing?

   (A) $\left(-\frac{1}{\sqrt{2}}, \infty\right)$

   (B) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

   (C) $(0, \infty)$

   (D) $(-\infty, 0)$

   (E) $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$

Part B. Graphing calculator allowed.

6. Let f be the function with derivative given by $f'(x) = \sin(x^2 + 1)$. How many relative extrema does f have on the interval $2 < x < 4$?
   (A) One (B) Two (C) Three (D) Four (E) Five

7. For all x in the closed interval [2, 5], the function f has a positive first derivative and a negative second derivative. Which of the following could be a table of values for f?

(A)

x	$f(x)$
2	7
3	9
4	12
5	16

(B)

x	$f(x)$
2	7
3	11
4	14
5	16

(C)

x	$f(x)$
2	16
3	12
4	9
5	7

(D)

x	$f(x)$
2	16
3	14
4	11
5	7

(E)

x	$f(x)$
2	16
3	13
4	10
5	7